Fin efficiency of annular fins made of two materials

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Abstract

The efficiency of two-material fins is presented. The analytical solution is compared to the expression obtained by Gardner for one-material fins. We conclude that Gardner's expression can be used for a two-material fin, by changing only the value of one parameter. The efficiency of a galvanized fin may be considerably increased by the coating. The ratio of the efficiency of the coated fin to the efficiency of the basic fin may increase by a factor of two for thin fins having a large fin tip radius to base radius ratio. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

It is well known that the use of extended surfaces increases the heat transfer between a base surfaces and a fluid [1]. The efficiency of different shapes of fins has already been established [2,3]. But the proposed equations do not allow a computation of the temperature distribution and of the efficiency of fins made of two materials. These kind of fins are encountered in galvanized steel heat exchangers. In this paper, we develop an expression for the temperature distribution and efficiency of such fins, taking into account the effect of the thickness of the fin.

Fig. 1 gives a schematic view of such a fin.

2. Theoretical development

The study of the temperature distribution is based upon the following common assumptions:

1. the heat transfer rate is steady;
2. the fin materials are homogeneous and isotropic;
3. the thermal conductivities, λ, of the fin are constant;
4. there are no heat sources in the fin;
5. the heat transfer can be calculated with Newton's law;
6. the heat transfer coefficient, h, is the same all over the fin surface;
7. the temperature, T_f, of the surrounding fluid is uniform;
8. the heat transferred through the fin tip is negligible;
9. the temperature at the root of the fin is known;
10. the contact resistance is negligible.

To simplify the presentation of the equations governing the studied phenomena, it is necessary to introduce the temperature differences $\theta_i = T_i(r, z) - T_f$. It is also possible to write the temperature distribution in one half of the fin (for $z \geq 0$).

The heat propagation in the central material of the fin may be written as Eq. (1):

$$\frac{\partial^2 \theta_1}{\partial z^2} + \frac{\partial^2 \theta_1}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \theta_1}{\partial r} = 0. \tag{1}$$

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Nomenclature

A	integration variable
a	integration variable
b	integration variable
B	integration variable
c	thickness
h	convection coefficient
I_n	modified Bessel function of the first kind and of order n
K_n	modified Bessel function of the second kind and of order n
q	heat flux
r	radius
R	function of r
T	temperature
z	abscissa
Z	function of z.

Greek symbols

δ	temperature distribution parameter for a one-material fin
δ_2	temperature distribution parameter for a two-material fin
η	fin efficiency
θ	temperature difference
λ	conductivity.

Subscripts

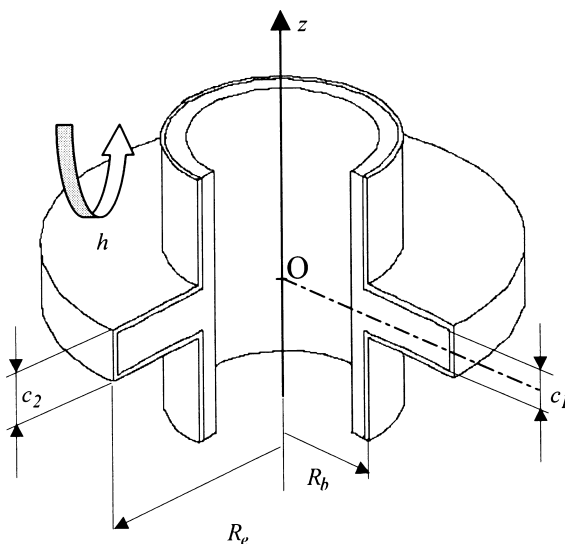|	
b	base of the fin
e	extremity of the fin
f	fluid
i	index of the material.

Fig. 1. Schematic view of a coated fin.

The heat propagation in the outer material of the fin may be written as Eq. (2):

$$\frac{\partial^2 \theta_2}{\partial z^2} + \frac{\partial^2 \theta_2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \theta_2}{\partial r} = 0. \tag{2}$$

At the interface of the two materials, the temperature in the central material is equal to the temperature in the outer material [Eq. (3)].

$$\theta_1 \left(r, \ z = \frac{c_1}{2} \right) = \theta_2 \left(r, \ z = \frac{c_1}{2} \right). \tag{3}$$

Likewise, the heat fluxes are equal at the interface of the two materials [Eq. (4)].

$$-\lambda_1 \cdot \left(\frac{\partial \theta_1}{\partial z} \right)_{\left(r, \ z = \frac{c_1}{2} \right)} = -\lambda_2 \cdot \left(\frac{\partial \theta_2}{\partial z} \right)_{\left(r, \ z = \frac{c_1}{2} \right)}. \tag{4}$$

At the fin surface, the heat flux by convection equals the heat flux by conduction, which may be written as Eq. (5):

$$h \cdot \theta_2\left(r, \ z = \frac{c_2}{2}\right) + \lambda_2\left(\frac{\partial \theta_2}{\partial z}\right)_{\left(r, \ z = \frac{c_2}{2}\right)} = 0. \qquad (5)$$

We suppose now that each temperature difference is the product of a function of r and a function of z, Eqs. (6) and (7):

$$\theta_1 = R_1(r) \cdot Z_1(z) \qquad (6)$$

$$\theta_2 = R_2(r) \cdot Z_2(z). \qquad (7)$$

Eq. (3) may be written as Eq. (8):

$$R_1(r) \cdot Z_1\left(\frac{c_1}{2}\right) = R_2(r) \cdot Z_2\left(\frac{c_1}{2}\right). \qquad (8)$$

The separation of the variables technique increases the numbers of unknowns without increasing the number of boundary conditions. So, we have to arbitrarily choose new boundary conditions. Here, we can impose $Z_1(c_1/2) = Z_2(c_1/2)$. In this case Eq. (3) gives Eq. (9):

$$R_1(r) = R_2(r). \qquad (9)$$

Thus, only one unknown has to be considered: $R(r) = R_1(r) = R_2(r)$.

To find the equation which will allow us to calculate $R(r)$, an energy balance is written in the fin at radius r. The conduction heat flux crossing a section at this arbitrary distance r may be written as the sum of the heat flux in the inner material and in the outer material [Eq. (10)]:

$$q = -\int_0^{c_1/2} \lambda_1 \cdot \frac{\partial \theta_1}{\partial r} \cdot 2 \cdot \pi \cdot r \cdot dz$$
$$- \int_{c_1/2}^{c_2/2} \lambda_2 \cdot \frac{\partial \theta_2}{\partial r} \cdot 2 \cdot \pi \cdot r \cdot dz. \qquad (10)$$

This heat flux only comes from energy convected into the fin between radius r and the outer radius of the fin. This may be written as Eq. (11):

$$q = \int_r^{R_e} h \cdot \theta_2\left(r, \ z = \frac{c_2}{2}\right) \cdot 2 \cdot \pi \cdot r \cdot dr. \qquad (11)$$

Combining Eqs. (10) and (11), introducing Eq. (9), taking the derivative of the combined equation, and rearranging the results, we obtain Eq. (12):

$$\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \cdot \frac{dR(r)}{dr}$$
$$- \frac{h \cdot Z_2\left(\frac{c_2}{2}\right)}{\lambda_1 \cdot \int_0^{c_1/2} Z_1(z) \cdot dz + \lambda_2 \cdot \int_{c_1/2}^{c_2/2} Z_2(z) \cdot dz} \qquad (12)$$
$$\cdot R(r) = 0.$$

This shows that R is a combination of the Bessel functions. We write the solution as Eq. (13):

$$R(r) = A \cdot I_0(\delta_2 \cdot r) + B \cdot K_0(\delta_2 \cdot r) \qquad (13)$$

with:

$$\delta_2^2 = \frac{h \cdot Z_2\left(\frac{c_2}{2}\right)}{\lambda_1 \cdot \int_0^{c_1/2} Z_1(z) \cdot dz + \lambda_2 \cdot \int_{c_1/2}^{c_2/2} Z_2(z) \cdot dz}. \qquad (14)$$

Note that Eq. (13) also represents the temperature distribution in one-material fins; the only difference is that the parameter used by Gardner ($\sqrt{2 \cdot h/\lambda \cdot c}$) has to be replaced by δ_2.

As the efficiency of a fin is defined as the ratio of the actual convection flux to the convection flux the fin would have transmitted if the outer surface were at the temperature of the foot of the fin, we may write the efficiency as Eq. (15):

$$\eta = \frac{\int_{R_b}^{R_e} h \cdot Z_2\left(\frac{c_2}{2}\right) \cdot R(r) \cdot 2 \cdot \pi \cdot r \cdot dr}{\int_{R_b}^{R_e} h \cdot Z_2\left(\frac{c_2}{2}\right) \cdot R(R_b) \cdot 2 \cdot \pi \cdot r \cdot dr}$$
$$= \frac{\int_{R_b}^{R_e} R(r) \cdot 2 \cdot \pi \cdot r \cdot dr}{\int_{R_b}^{R_e} R(R_b) \cdot 2 \cdot \pi \cdot r \cdot dr}. \qquad (15)$$

The efficiency depends only on the R function. Hence, it depends only on the parameter δ_2 or $\sqrt{2 \cdot h/\lambda \cdot c}$. This shows that it is possible to calculate the fin efficiency for two-material fins using the curves that have been established by Gardner for one-material fins, replacing the parameter δ by δ_2. We now need to develop an expression for δ_2.

Introducing the R and Z functions in Eqs. (1) and (2), we find:

$$\frac{d^2 Z_1(z)}{dz^2} \cdot R(r) + Z_1(z) \cdot \left(\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr}\right) = 0 \qquad (16)$$
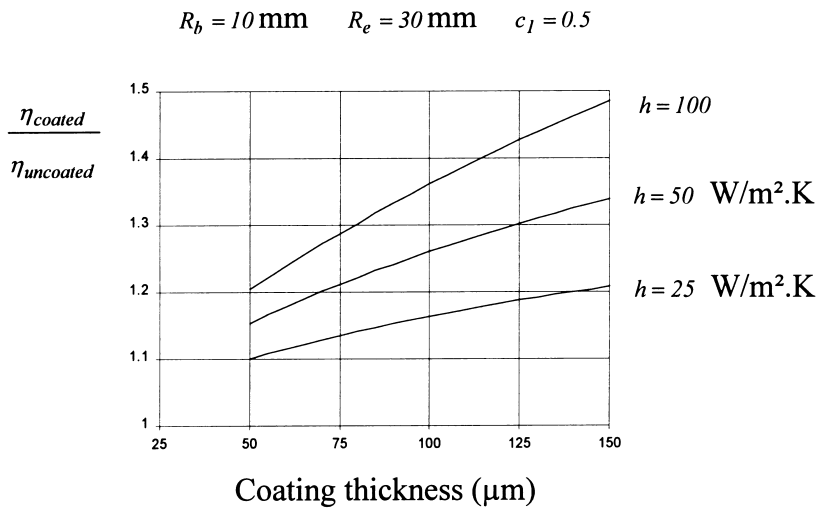
$$R_b = 10\,\text{mm} \qquad R_e = 30\,\text{mm} \qquad c_1 = 0.5$$

Fig. 2. Influence of the convection coefficient on the efficiency ratio.

$$\frac{\mathrm{d}^2 Z_2(z)}{\mathrm{d}z^2} \cdot R(r) + Z_2(z) \cdot \left(\frac{\mathrm{d}^2 R(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R(r)}{\mathrm{d}r} \right) = 0. \quad (17)$$

A substitution from Eq. (12) can be made for the two bracketed terms. Then, substitutions in Eqs. (16) and (17) result in Eqs. (18) and (19):

$$\frac{\mathrm{d}^2 Z_1(z)}{\mathrm{d}z^2} + \delta_2^2 \cdot Z_1(z) = 0 \tag{18}$$

$$\frac{\mathrm{d}^2 Z_2(z)}{\mathrm{d}z^2} + \delta_2^2 \cdot Z_2(z) = 0. \tag{19}$$

It is possible to write general expressions for $Z_1(z)$ and $Z_2(z)$:

$$Z_1(z) = a_1 \cdot \cos(\delta_2 \cdot z) + b_1 \cdot \sin(\delta_2 \cdot z)$$

$$Z_2(z) = a_2 \cdot \cos(\delta_2 \cdot z) + b_2 \cdot \sin(\delta_2 \cdot z).$$

The symmetry condition on the symmetry plane of the fin gives $b_1 = 0$. These expressions, combined with Eq. (9), can be used to rewrite Eqs. (3)–(5). From these equations we can deduce:

$$R_b = 10\,\text{mm} \quad c_1 = 0.5\,\text{mm} \quad h = 50\,\text{W/m}^2.\text{K}$$
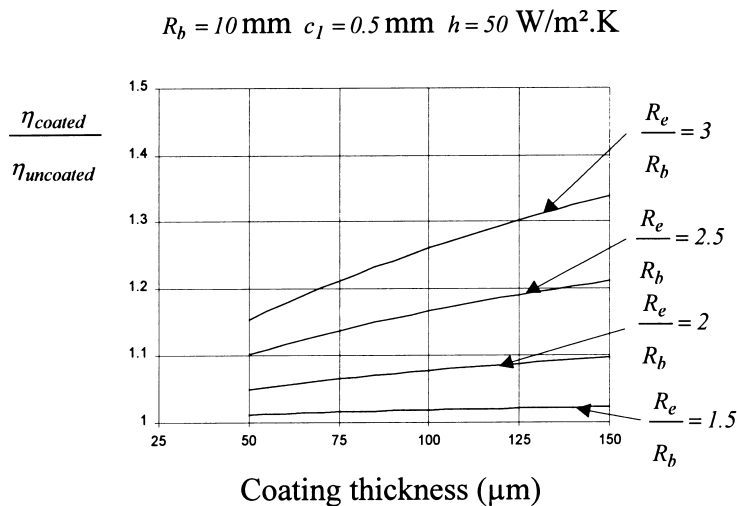

Fig. 3. Influence of the radius ratio on the efficiency ratio.

$$a_2 = a_1 \cdot \frac{\lambda_2 + \lambda_1 \cdot \mathrm{tg}^2\left(\delta_2 \cdot \frac{c_1}{2}\right)}{\lambda_2 \cdot \left[1 + \mathrm{tg}^2\left(\delta_2 \cdot \frac{c_1}{2}\right)\right]} \tag{20}$$

$$b_2 = a_1 \cdot \frac{\lambda_2 - \lambda_1}{2 \cdot \lambda_2} \cdot \sin(\delta_2 \cdot c_1) \tag{21}$$

$$\delta_2 = \frac{h}{\lambda_2} \cdot \frac{\lambda_2 \cdot \cos\left(\delta_2 \cdot \frac{c_1}{2}\right) \cdot \cos\left(\delta_2 \cdot \frac{c_2 - c_1}{2}\right) - \lambda_1 \cdot \sin\left(\delta_2 \cdot \frac{c_1}{2}\right) \cdot \sin\left(\delta_2 \cdot \frac{c_2 - c_1}{2}\right)}{\lambda_2 \cdot \cos\left(\delta_2 \cdot \frac{c_1}{2}\right) \cdot \sin\left(\delta_2 \cdot \frac{c_2 - c_1}{2}\right) + \lambda_1 \cdot \sin\left(\delta_2 \cdot \frac{c_1}{2}\right) \cdot \cos\left(\delta_2 \cdot \frac{c_2 - c_1}{2}\right)}. \tag{22}$$

Eq. (22) gives an implicit equation in δ_2, but the value of δ_2 may be approximated, for thin fins with a thin coating, by noting $\cos(\varphi) \to 1$ and $\sin(\varphi) \to \varphi$, and $c_1 \cdot (c_2 - c_1) \to 0$. We obtain Eq. (23):

$$\delta_2 = \sqrt{\frac{2 \cdot h}{\lambda_2 \cdot (c_2 - c_1) + \lambda_1 \cdot c_1}}. \tag{23}$$

We can note that when $\lambda_1 = \lambda_2$, $c_1 = 0$, or $c_2 = c_1$ (one-material fin), we find that Eq. (23) reduces to the traditional expression of δ:

$$\delta = \sqrt{\frac{2 \cdot h}{\lambda \cdot c}}. \tag{24}$$

The separation of the variables technique has led us to write four second-order differential equations. Hence, we have to determine eight unknowns. We have already transcribed four boundary conditions. The last boundary conditions we know are: the heat transferred through the outermost edge of the fin is negligible, and the temperature at the base of the fin is imposed; these boundary conditions are used to determine A and B. This makes only six boundary conditions. So, it is possible to arbitrarily choose two more boundary conditions. We have already written $Z_1(c_1/2) = Z_2(c_1/2)$. The last boundary condition we choose is:

$$Z_2\left(\frac{c_2}{2}\right) = 1$$

or:

$$a_1 = \frac{\lambda_2}{\lambda_2 \cdot \cos\left(\delta_2 \cdot \frac{c_1}{2}\right) \cdot \cos\left(\delta_2 \cdot \frac{c_2 - c_1}{2}\right) - \lambda_1 \cdot \sin\left(\delta_2 \cdot \frac{c_1}{2}\right) \cdot \sin\left(\delta_2 \cdot \frac{c_2 - c_1}{2}\right)}. \tag{25}$$

Then we are able to give the expression of the R function:

$$R(r) = \theta\left(R_{\mathrm{b}}, \frac{c_2}{2}\right) \cdot \frac{K_1(\delta_2 \cdot R_{\mathrm{e}}) \cdot I_0(\delta_2 \cdot r) + I_1(\delta_2 \cdot R_{\mathrm{e}}) \cdot K_0(\delta_2 \cdot r)}{K_1(\delta_2 \cdot R_{\mathrm{e}}) \cdot I_0(\delta_2 \cdot R_{\mathrm{b}}) + I_1(\delta_2 \cdot R_{\mathrm{e}}) \cdot K_0(\delta_2 \cdot R_{\mathrm{b}})}. \tag{26}$$

We can finally give the expression of the efficiency of the fin:

$$\eta = \frac{2 \cdot R_{\mathrm{b}}}{\delta_2 \cdot (R_{\mathrm{e}}^2 - R_{\mathrm{b}}^2)} \cdot \frac{I_1(\delta_2 \cdot R_{\mathrm{e}}) \cdot K_1(\delta_2 \cdot R_{\mathrm{b}}) - K_1(\delta_2 \cdot R_{\mathrm{e}}) \cdot I_1(\delta_2 \cdot R_{\mathrm{b}})}{I_1(\delta_2 \cdot R_{\mathrm{e}}) \cdot K_0(\delta_2 \cdot R_{\mathrm{b}}) + K_1(\delta_2 \cdot R_{\mathrm{e}}) \cdot I_0(\delta_2 \cdot R_{\mathrm{b}})}. \tag{27}$$

3. Application

Consider fins made of steel and coated by zinc. For the steel suppose [1] $\lambda_1 = 50$ W m^{-1} K^{-1} and for the zinc $\lambda_2 = 111$ W m^{-1} K^{-1}. Coating thicknesses range from 50 to 150 μm. Fins typically have a thickness between 0.2 and

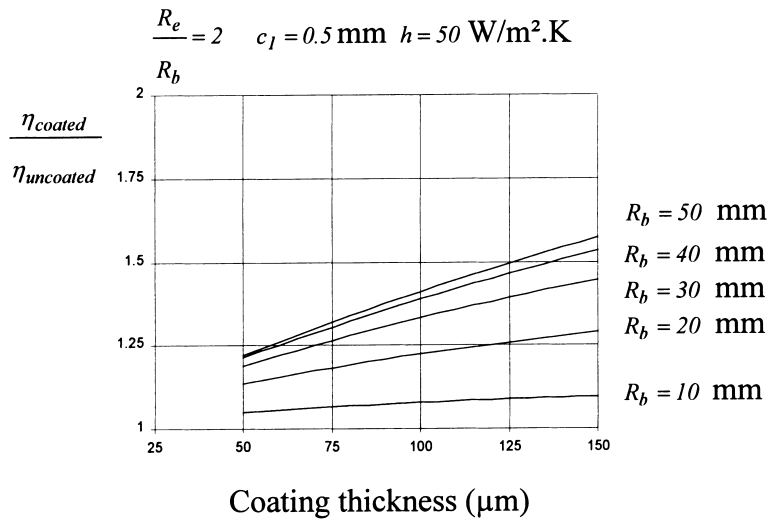$$\frac{R_e}{R_b} = 2 \quad c_l = 0.5 \text{ mm} \quad h = 50 \text{ W/m}^2.\text{K}$$

Fig. 4. Influence of the fin base radius on the efficiency ratio.

1 mm. The diameter of the tubes that support the fins generally ranges from 10 to 50 mm, and the outer diameter of the fin generally ranges from 1.5 to 3 times the tube diameter. The convection coefficient varies from 25 to 150 W m^{-2} K^{-1}.

Figs. 2–5 compare the efficiency of coated fins to the efficiency of uncoated fins for various geometrical characteristics, convection coefficients, and coating thicknesses.

As can be seen, the thicker the coating, the more efficient the fin. When the convection coefficient increases (Fig. 2), when the radius ratio increases (Fig. 3), or when the fin base radius increases (Fig. 4), the temperature gradient in the fin increases. Likewise, the more uniform the temperature, the higher the efficiency of the fin. So, the role of the coating is clear: it increases the apparent conductivity of the fin, reducing the temperature gradient and increasing the fin efficiency. With a thicker core fin, the effect of the zinc on the apparent conductivity is reduced. Hence, the thicker the basic fin, the lower is the gain in the fin efficiency (Fig. 5).

We have to note that, for all these examples, the difference between the solution of the implicit Eq. (22), which is obtained using Mathematica, and the simplified value obtained by Eq. (23) is less than 0.01%.
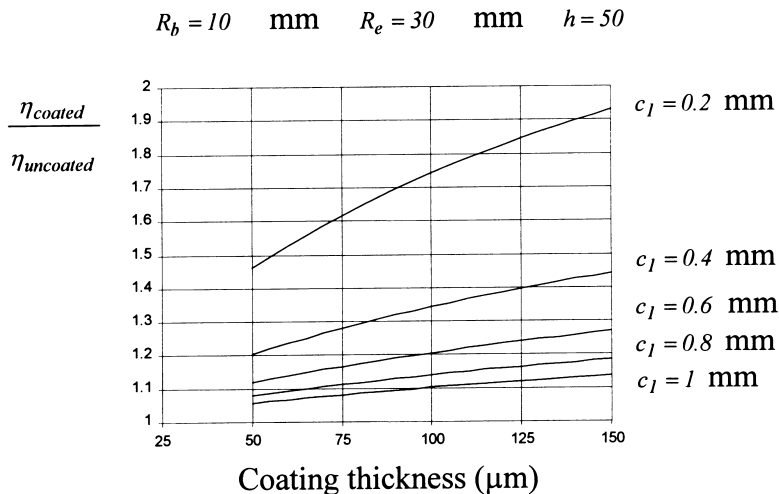
$$R_b = 10 \text{ mm} \quad R_e = 30 \text{ mm} \quad h = 50$$

Fig. 5. Influence of the thickness of the core material on the efficiency ratio.

Hence, the simplified equation can be used with confidence.

4. Conclusion

Computation of the temperature distribution in annular fins made of two materials has been performed. An expression of the efficiency of fins made of two materials has been developed. A simplified expression of the main parameter is given and this approximation is justified for ordinary fin geometries.

We have shown that, by example, a coating with a higher thermal conductivity increases the efficiency of fins. The coating is more important for fins that have a low efficiency when they are uncoated (large height fins, thin fins). The efficiency ratio may be increased by a factor of two.

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